Fore-and-Aft Translations from FOL into Coherent Logic

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Abstract

This note describes an intriguing method for translating FOL into the *geolog* language for coherent logic. The method systematically attempts to translate occurrences of formal implication in FOL inputs into coherent logic rules containing implication in a "forward" direction, rather than expressing formal implication using a disjunction, which involves a "splitting" or "case" translation. The advantage of forward translation is a reduced search space when the resulting rules are used to compute consequences: The forward rules maintain the current branch deduction, whereas a splitting translation promotes new branches.

1 Background information and motivating examples

An elegant direct method for a translation from FOL (*first-order logic*) into coherent logic was outlined in [1]. The method consists of encoding the semantic tableaux proof method into a set of coherent rules, that is, a *geolog theory* that can be computed by a *Skolem Machine* [2] . See reference [3] for more information about FOL translation, another characterization of the basic translation method, and an important detailed explanation for why these translations yield *correct proof objects*, as opposed to other FOL translation approaches which do not.

Andrew Polonsky has worked on promising *heuristic* translation methods that "favor" forward translation, using specific contrapositive forms of disjunctive coherent rules (discussed briefly in [3]).

This note specifies a *deterministic* algorithm for forward translation that arises from formal implication. In order to motivate this approach, we present several telling examples.

Example 1. Supose that $\phi = ((a \vee b) \to c) \to d$ is an axiom. The ϕ is converted to geolog rules, as follows:

$$true \Rightarrow \overrightarrow{((a \vee b) \to c) \to d}.$$

$$\overrightarrow{((a \vee b) \to c) \to d}, \overleftarrow{((a \vee b) \to c)} \Rightarrow \overrightarrow{d}.$$

$$\overleftarrow{\neg(a \vee b)} \Rightarrow \overleftarrow{((a \vee b) \to c)}.$$

$$\overleftarrow{c} \Rightarrow \overleftarrow{((a \vee b) \to c)}.$$

$$\overleftarrow{\neg a}, \overleftarrow{\neg b} \Rightarrow \overleftarrow{\neg(a \vee b)}.$$

$$not\_a \Rightarrow \overleftarrow{\neg a}.$$

$$not\_b \Rightarrow \overleftarrow{\neg b}.$$

$$c \Rightarrow \overleftarrow{c}.$$

$$\overrightarrow{d} \Rightarrow d.$$

In addition, we always have consistency axioms. Explicit negation of a predicate p is expressed here as not_p.

$$a, not\_a \Rightarrow false.$$

$$b, not\_b \Rightarrow false.$$

$$c, not\_c \Rightarrow false.$$

$$d, not\_d \Rightarrow false.$$

The *fore operator* $\overleftarrow{\ \ }$ is *tried* whenever formal implication is encountered. The trial fails when an exceptional case is encountered while trying to proceed in the fore direction. In this example *fore translation* goes smoothly, without exception. The *aft operator* $\overrightarrow{\ \ }$ always applies. Example 3 below will illustrate how fore translation can lead to an exceptional case, and the examples explain how to translate when an exception arise.

The next example illustrates some fore and aft translation patterns using quantified variables. Again, the fore translation goes smoothly.

Example 2. Supose that $\phi = (\forall x)(\neg p(x) \rightarrow (\exists y)(\exists z)q(x,u,z))$.

$$true \Rightarrow \overrightarrow{(\forall(x)(\neg p(x) \rightarrow (\exists y)(\exists z)q(x,u,z))}.$$

$$\overrightarrow{(\forall(x)(\neg p(x) \rightarrow (\exists y)(\exists z)q(x,u,z))}, dom(X) \Rightarrow \overrightarrow{\neg p(x) \rightarrow (\exists y)(\exists z)q(x,u,z)}(X).$$

$$\overrightarrow{\neg p(x) \rightarrow (\exists y)(\exists z)q(x,u,z)}(X), \overleftarrow{\neg p(x)}(X) \Rightarrow \overrightarrow{(\exists y)(\exists z)q(x,u,z)}(X).$$

$$not\_p(X) \Rightarrow \overleftarrow{\neg p(x)}(X).$$

$$\overrightarrow{(\exists y)(\exists z)q(x,u,z)}(X) \Rightarrow dom(Y), \overrightarrow{(\exists z)q(x,u,z)}(X,Y).$$

$$\overrightarrow{(\exists z)q(x,u,z)}(X,Y) \Rightarrow dom(Z), \overrightarrow{q(x,u,z)}(X,Y,Z).$$

$$\overrightarrow{q(x,u,z)}(X,Y,Z) \Rightarrow q(X,Y,Z).$$

The geolog rules are verbosely expressed in order to illustrate translation steps incrementally. Automated translators can skip or combine several steps without loss of generality. Also, the variable correspondences are presented in an intuitive fashion (e.g., formal variable x becomes X in the geolog rule).

The next example illustrates an exceptional case of fore translation. The corresponding (embedded) formal implication will have to be expressed using disjunction in a coherent rule.

Example 3. Supose that $\phi = (\forall x)((\forall y)p(x,y) \rightarrow q(x))$. Note that there is an embedded universal quantifier in the antecedent of the embedded formal implication. First, let us see how the exception to fore translation arises ...

$$true \Rightarrow \overrightarrow{(\forall x)((\forall y)p(x,y) \rightarrow q(x))}.$$

$$\overrightarrow{(\forall x)((\forall y)p(x,y) \rightarrow q(x))}, dom(X), \Rightarrow \overrightarrow{((\forall y)p(x,y) \rightarrow q(x))}(X).$$

$$\star \quad \overrightarrow{((\forall y)p(x,y) \rightarrow q(x))}(X), \overleftarrow{(\forall y)p(x,y)}(X) \Rightarrow \overrightarrow{q(x)}(X).$$

$$\textbf{EXCEPTIONAL}: \ldots ??? \Rightarrow \overleftarrow{(\forall y)p(x,y)}(X).$$

so $\star$ is replace by ...

$$\overrightarrow{((\forall y)p(x,y) \to q(x))}(X) \Rightarrow \overrightarrow{\neg(\forall y)p(x,y)}(X) \mid \overrightarrow{q(x)}(X).$$

and continue ...

$$\overrightarrow{\neg(\forall y)p(x,y)}(X) \Rightarrow \overrightarrow{(\exists y)\neg p(x,y)}(X).$$

$$\overrightarrow{(\exists y)\neg p(x,y)}(X) \Rightarrow dom(Y), \overrightarrow{\neg p(x,y)}(X,Y).$$

$$\overrightarrow{\neg p(x,y)}(X,Y) \Rightarrow not\_p(X,Y).$$

In this example, the attempt to use a fore translation for the embedded formal implication leads to an exception: There is no fore translation for a universal form. Thus the translation must be redone with aft translation.

2 Formal schemas defining fore-and-aft translations

Let us assume that the FOL theory has zero or more *axioms* of the form $\top \to \alpha$ (affirmation) and one *conjecture* of the form $\gamma \to \bot$ (denial).

The axioms and conjecture are translated into geolog rules. An axiom $\top \to \alpha$ is translated into the geolog rule $true \Rightarrow \overrightarrow{\alpha}$ (affirm axiom) together with translations for $\overrightarrow{\alpha}$. The conjecture $\gamma \to \bot$ is translated into geolog rule $true \Rightarrow \overrightarrow{\neg\gamma}$ (denial of conjecture) together with translations for $\overrightarrow{\neg\gamma}$.

The right-facing arrow $\dashrightarrow$ indicates the "aft" direction for translation. A so-called "fore" direction $\dashleftarrow$ will arise upon consideration of the detailed translations for the $\overrightarrow{\alpha}$ (axiom) or $\overrightarrow{\sim\gamma}$ (conjecture) forms, and this is what we now specify.

Fig. 1 specifies the *aft* translations. These are essential the same as those specified in Reference [1] except for schema (4). Notice that schema (4) is the only aft schema to introduce a *fore* translation. If the fore translation excounters an exception then one uses the aft schema to translate the form. Fig. 2 specifies the fore translations.

In the schemas, a sequence $\hat{X}$ always denotes an ordered sequence of variable arguments that arose *before* the current schema is supposed to apply. For convenience, we assume that the lower-case variable letters in the FOL wffs become upper-case when the corresponding variable gets elevated to arguments in the geolog rules.

(A1) $\overrightarrow{\phi \wedge \psi}(\hat{X}) \Rightarrow \overrightarrow{\phi}(\hat{X}), \overrightarrow{\psi}(\hat{X})$

(A2) $\overrightarrow{\phi \vee \psi}(\hat{X}) \Rightarrow \overrightarrow{\phi}(\hat{X}) \mid \overrightarrow{\psi}(\hat{X})$

(A3 a) $\overrightarrow{\neg(\phi \wedge \psi)}(\hat{X}) \Rightarrow \overrightarrow{\neg\phi}(\hat{X}) | \overrightarrow{\neg\psi}(\hat{X})$

b) $\overrightarrow{\neg(\phi \vee \psi)}(\hat{X}) \Rightarrow \overrightarrow{\neg\phi}(\hat{X}), \overrightarrow{\neg\psi}(\hat{X})$

c) $\overrightarrow{\neg\neg\phi} \Rightarrow \overrightarrow{\phi}$

d) $\overrightarrow{\neg(\phi \rightarrow \psi)}(\hat{X}) \Rightarrow \overrightarrow{\phi}(\hat{X}), \overrightarrow{\neg\psi}(\hat{X})$

e) $\overrightarrow{\neg(\forall \hat{x})\phi}(\hat{Y}) \Rightarrow \overrightarrow{(\exists \hat{x})\neg\phi}(\hat{Y})$

f) $\overrightarrow{\neg(\exists \hat{x})\phi}(\hat{Y}) \Rightarrow \overrightarrow{(\forall \hat{x})\neg\phi}(\hat{Y})$

g) $\overrightarrow{\neg a(\hat{x})} \Rightarrow not\_a(\hat{X})$ when a is atomic

(A4) $\overrightarrow{\phi \rightarrow \psi}(\hat{X}), \overleftarrow{\phi}(\hat{X}) \Rightarrow \overrightarrow{\psi}(\hat{X})$ if translation of $\overleftarrow{\phi}$ is *NOT EXCEPTIONAL*
$\overrightarrow{\phi \rightarrow \psi}(\hat{X}) \Rightarrow \overrightarrow{\neg\phi} \mid \overrightarrow{\psi}(\hat{X})$ otherwise

(A5) $\overrightarrow{(\forall x)\phi}(\hat{Y}), dom(X) \Rightarrow \overrightarrow{\phi}(\hat{Y}, X)$

(A6) $\overrightarrow{(\exists x)\phi}(\hat{Y}) \Rightarrow dom(X), \overrightarrow{\phi}(\hat{Y}, X)$

(A7) $\overrightarrow{p(\hat{t})}(\hat{X}) \Rightarrow p(\hat{T})$ for predicate p, where terms $\hat{T}$ are substitution results for terms $\hat{t}$

When a fore translation is encountered at (A4), it either completes without exception, and the rules it produces using translations from Fig. 2 are included in the converted theory, or an exception arises and the corresponding rules are discarded, and then the aft version of (A4) is used to obtained the converted rules.

Figure 1: **Aft** *translations*

(F1) $\overleftarrow{\phi}(\hat{X}), \overleftarrow{\psi}(\hat{X}) \Rightarrow \overleftarrow{\phi \wedge \psi}(\hat{X})$

(F2) $\overleftarrow{\phi}(\hat{X}) \Rightarrow \overleftarrow{\phi \vee \psi}(\hat{X})$
$\overleftarrow{\psi}(\hat{X}) \Rightarrow \overleftarrow{\phi \vee \psi}(\hat{X})$

(F3 a) $\overleftarrow{\neg\phi}(\hat{X}) \Rightarrow \overleftarrow{\neg(\phi \wedge \psi)}(\hat{X})$
$\overleftarrow{\neg\psi}(\hat{X}) \Rightarrow \overleftarrow{\neg(\phi \wedge \psi)}(\hat{X})$

b) $\overleftarrow{\neg\phi}(\hat{X}), \overleftarrow{\neg\psi}(\hat{X}) \Rightarrow \overleftarrow{\neg(\phi \vee \psi)}(\hat{X})$

c) $\overleftarrow{\phi} \Rightarrow \overleftarrow{\neg\neg\phi}$

d) $\overleftarrow{\phi}(\hat{X}), \overleftarrow{\neg\psi}(\hat{X}) \Rightarrow \overleftarrow{\neg(\phi \rightarrow \psi)}(\hat{X})$

e) **EXCEPTIONAL** : ...??? $\Rightarrow \overleftarrow{\neg(\exists\hat{x})\phi}(\hat{Y})$

f) $\overleftarrow{(\exists x)\neg\phi}(\hat{Y}) \Rightarrow \overleftarrow{\neg(\forall\hat{x})\phi}(\hat{Y})$

g) $not\_a(\hat{X}) \Rightarrow \overleftarrow{\neg a(\hat{x})}$ when a is atomic

(F4) $\overleftarrow{\neg\phi}(\hat{X}) \Rightarrow \overleftarrow{\phi \rightarrow \psi}(\hat{X})$.
$\overleftarrow{\psi}(\hat{x}) \Rightarrow \overleftarrow{\phi \rightarrow \psi}(\hat{X})$

(F5) **EXCEPTIONAL** : ...??? $\Rightarrow \overleftarrow{(\forall\hat{x})\phi}(\hat{Y})$

(F6) $\overleftarrow{\phi}(\hat{Y}, X) \Rightarrow \overleftarrow{(\exists\hat{x})\phi}(\hat{Y})$

(F7) $p(\hat{T}), dom(\hat{X}) \Rightarrow \overleftarrow{p(\hat{t})}((\hat{T})$ for predicate p, where terms $\hat{T}$ are substitution results for terms $\hat{t}$, and sequence $\hat{X}$ consiste of variables in $\hat{T}$ which do not occur in $p(\hat{t})$

Figure 2: **Fore** *translations*

In essence, the exceptional cases involve the obligation to provide a fore translation for a universally quantified wff. There is no simple coherent translation for this case. It is clear that both translation methods are sound. We illustrate the required complication in rule (F7) using another example.

Example 4. Suppose that the axiom is $\top \rightarrow (\forall x)(\forall y)(p(x) \rightarrow q(x,y))$ and notice that the antecedent of the formal implication does not contain the universal variable y.

$$true \Rightarrow \overrightarrow{(\forall x)(\forall y)(p(x) \rightarrow q(x,y))}.$$

$$\overrightarrow{(\forall x)(\forall y)(p(x) \rightarrow q(x,y))}, dom(X), dom(Y) \Rightarrow \overrightarrow{p(x) \rightarrow q(x,y)}(X,Y).$$

$$\overrightarrow{p(x) \rightarrow q(x,y)}(X,Y), \overleftarrow{p(x)}(X,Y) \Rightarrow \overrightarrow{q(x,y)}(X,Y).$$

$$\star \; p(X), dom(Y) \Rightarrow \overleftarrow{p(x)}(X,Y).$$

$$\overrightarrow{q(x,y)}(X,Y) \Rightarrow q(x,y).$$

Notice at $\star$ how $dom(Y)$ is needed in the antecedent to universally quantify Y; otherwise the occurrence in the consequent would be interpreted as an existential variable.

Theorem. *If $\alpha = (\forall \hat{x})[\Phi \rightarrow (\exists \hat{y}_1)\psi_1 \vee \ldots \vee (\exists \hat{y}_n)\psi_n]$ is a coherent form FOL axiom then the terse form of the aft translation $\overrightarrow{\alpha}$ is the corresponding geolog rule $\Phi, dom(\hat{X}) \Rightarrow dom(\hat{Y}_1), \psi_1 \mid \ldots \mid dom(\hat{Y}_n), \psi_n$, formed by replacing the quantifiers with $dom(-)$'s. ($\Phi, \Psi_1, \ldots, \Psi_n$ are conjunctions of literals.)*

Exercise 1. For each of the translations in Examples 1-4, annotate each step of the translation with the appropriate fore (F#) or aft schema (A#) number.

Exercise 2. Using the *colog1.8* prover, generate terse translations for each of the Examples 1-4.

Exercise 3. Appendix A lists a FOL version of the dpe problem. Convert this into the input form required by *GeologUI* and *colog* provers and save in a file `dpe.fol`. Compare the outcomes produced for `dpe.fol` using two approaches: a) load into *GeologUI*, using terse form, but do not preserve coherent axioms, and b) load into *colog*, using terse form. What is suggested by this experiment regarding the preservation of coherent axioms?

3 Completeness

In order to illustrate the issues involved with completeness of the fore and aft translation schemas, consider the FOL theory ...

$$\top \rightarrow \neg r$$

$$\top \rightarrow (p \vee q \rightarrow r)$$

$$\neg q \rightarrow \bot$$

Here is an aft-only translation of the FOL theory ...

$r, not\_r \Rightarrow false.$
$p, not\_p \Rightarrow false.$
$q, not\_q \Rightarrow false.$

$true \Rightarrow \overrightarrow{\neg\neg q}.$
$\overrightarrow{\neg\neg q} \Rightarrow q.$

$true \Rightarrow \overrightarrow{\neg r}.$
$\overrightarrow{\neg r} \Rightarrow not\_r.$

$true \Rightarrow \overrightarrow{p \vee q \rightarrow r}.$
$\overrightarrow{p \vee q \rightarrow r} \Rightarrow \overrightarrow{\neg(p \vee q)} \mid r.$
$\overrightarrow{\neg(p \vee q)} \Rightarrow not\_p, not\_q.$

And here is a fore-and-aft translation ...

$r, not\_r \Rightarrow false.$
$p, not\_p \Rightarrow false.$
$q, not\_q \Rightarrow false.$

$true \Rightarrow \overrightarrow{\neg\neg q}.$
$\overrightarrow{\neg\neg q} \Rightarrow q.$

$true \Rightarrow \overrightarrow{\neg r}.$
$\overrightarrow{\neg r} \Rightarrow not\_r.$

$true \Rightarrow \overrightarrow{p \vee q \rightarrow r}.$
$\overrightarrow{p \vee q \rightarrow r}, \overleftarrow{p \vee q} \Rightarrow r.$
$p \Rightarrow \overleftarrow{p \vee q}$
$q \Rightarrow \overleftarrow{p \vee q}$

A proof tree for the aft-only translation is shown in Fig.3, and a proof tree for the fore-and-aft translation is shown in Fig.4

Any geolog tree that can be generated using an aft-only translation of a FOL theory using schema (A4) could also be constructed using the fore version of (A4), so long as the fore version does not meet an exceptional translation (in which case the aft version is used anyway). This is illustrated in our example: notice how the left branch in Fig. 3 could be constructed using contrapositive inferences from above node 5 in Fig. 4 using all available fore rules.

Reference [1] argues that a translation scheme which is equivalent to our aft-only method of translation is complete. Thus, if the FOL theory is valid then the aft-translated coherent theory will yield a proof. Based upon the observations above, the fore-and-aft version will also yield a proof, but often requiring fewer steps. This plausibility argument needs to be replaced by a more formal proof.

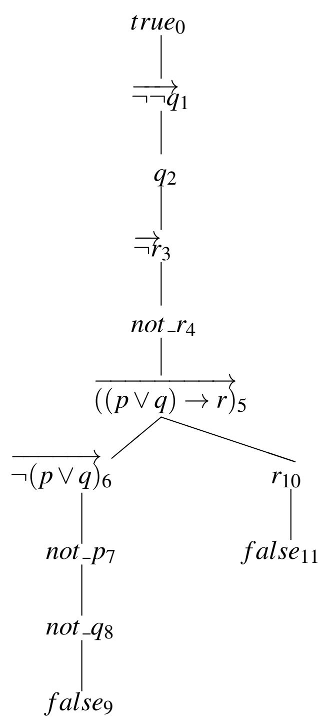

Figure 3: Aft-only proof tree

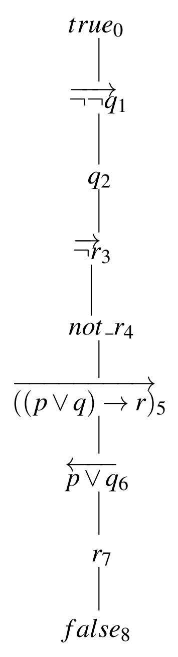

Figure 4: Fore-and-aft proof tree

References

[1] M. Bezem and T. Coquand, Automating Coherent Logic. In G. Sutcliffe and A. Voronkov, editors, *Proc. LPAR-12*, LNCS 3825, 2005, pages 246-260.

[2] John Fisher and Marc Bezem, Skolem Machines, *Fundamenta Informaticae*, 91 (1) 2009, pp.79-103.

[3] Andrew Polonsky and Marc Bezem, Proof Objects for Logical Translations, Proc. The 1st Coq Workshop, Munich, Germany 21 August, 2009, pages 49-61 `http://coq.inria.fr/files/coq-workshop-TUM-I0919.pdf`.

A FOL theory dpe

axioms:

$dom(a) \wedge dom(b) \wedge dom(c) \wedge re(a,b) \wedge re(a,c)$

$(\forall x)(dom(X) \to X = X)$

$(\forall x)(\forall y)(x = y \to y = x)$

$(\forall x)(\forall y)(\forall z)(x = y \wedge re(y,z) \to re(x,z))$

$(\forall x)(\forall y)(x = y \to re(x,y))$

$(\forall x)(\forall y)(r(x,y) \to re(x,y))$

$(\forall x)(\forall y)(re(x,y) \to (x = y \vee r(x,y))$

$(\forall x)(\forall y)(\forall y)(r(x,y) \wedge r(x,z) \to (\exists u)(r(y,u) \wedge r(z,u))$

conjecture:

$(\exists x)(re(b,X) \wedge re(c,X)))$

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